

Approximation of Invariant Measures for Regime-Switching Diffusions

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Abstract

In this paper, we are concerned with long-time behavior of Euler-Maruyama schemes associated with a range of regime-switching diffusion processes. The key contributions of this paper lie in that existence and uniqueness of numerical invariant measures are addressed (i) for regime-switching diffusion processes with finite state spaces by the Perron-Frobenius theorem if the “averaging condition” holds, and, for the case of reversible Markov chain, via the principal eigenvalue approach provided that the principal eigenvalue is positive; (ii) for regime-switching diffusion processes with countable state spaces by means of a finite partition method and an M-Matrix theory. We also reveal that numerical invariant measures converge in the Wasserstein metric to the underlying ones. Several examples are constructed to demonstrate our theory.

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1 Introduction

For a regime-switching diffusion process (RSDP), we mean a diffusion process in a random environment characterized by a Markov chain. The state vector of an RSDP is a pair (X_t, Λ_t) . Here $\{X_t\}_{t \geq 0}$ satisfies a stochastic differential equation (SDE)

$$dX_t = b(X_t, \Lambda_t)dt + \sigma(X_t, \Lambda_t)dW_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^n, \Lambda_0 = i \in \mathbb{S}, \quad (1.1)$$

and $\{\Lambda_t\}_{t \geq 0}$ denotes a continuous-time Markov chain with the state space $\mathbb{S} := \{1, 2, \dots, N\}$, $1 \leq N \leq \infty$, and the transition rules specified by

$$\mathbb{P}(\Lambda_{t+\Delta} = j | \Lambda_t = i) = \begin{cases} q_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + q_{ii}\Delta + o(\Delta), & i = j. \end{cases} \quad (1.2)$$

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Various quantities of (1.1) will be given in the next section. RSDPs have considerable applications in e.g. control problems, storage modeling, neural activity, biology and mathematical finance (see e.g. monographs [15, 31]). Pinsky and Scheutzow [19] showed that the overall system (X_t^x, Λ_t^i) need not to be positive recurrence (resp. transience) even when each subsystem is positive recurrence (resp. transience), and Mao and Yuan revealed in [15, Example 5.45, p.223] that (X_t^x, Λ_t^i) is stable although some of the subsystems are not. So, in some cases, the dynamical behavior of RSDPs may be markedly different from diffusion processes without regime switchings. So far, the works on RSDPs have included ergodicity [3, 9, 22, 21, 26], stability [15, 21, 27, 31], recurrence and transience [18, 19, 20, 31], invariant densities [1, 2], hypoellipticity [1, 4], and so forth.

Since solving RSDPs is still a challenging task, numerical schemes and/or approximation techniques have become one of the viable alternatives, where [15, 31] are concerned with finite-time (strong or weak) convergence while [10, 15] are devoted to long-time behavior of numerical schemes. For more details on numerical analysis of diffusion processes without regime switching, please refer to the monograph [12]. Also, approximations of invariant measures for stochastic dynamical systems have attracted much attention, see e.g. Mattingly et al. [16] via a Poisson equation, Talay [24] through the Kolmogorov equation, and Bréhier [5] by means of the Malliavin calculus. For the counterpart associated with Euler-Maruyama (EM) algorithms with constant/decreasing stepsize of RSDPs, we refer to Mao et al. [14] and Yuan and Mao [29] adopting the M-matrix theory, and Yin and Zhu [31] utilizing the weak convergence method, where RSDPs therein enjoy finite state spaces. Moreover, sufficient conditions imposed in [14, 29, 31] to guarantee existence of numerical invariant measures are irrelevant to stationary distributions of the continuous-time Markov chains that can accommodate a set of possible regimes.

Motivated by [14, 29, 31], in this paper we are also interested in numerical approximation of invariant measure for RSDP (1.1) and (1.2). In particular, we are concerned with the following questions:

- (i) Under what conditions, will the discrete-time semigroup generated by EM scheme admit a unique invariant measure?
- (ii) Will the numerical invariant measure, if it exists, converge in some metric to the underlying one?

In this paper, we shall answer the questions above one-by-one in several cases.

Throughout the paper, we stipulate $N < \infty$ in Section 2-4, and $N = \infty$ in Section 5. The content of this paper is arranged as follows. In Section 2, by the Perron-Fronenius theorem, we discuss existence and uniqueness of invariant measure for semigroup generated by (X_t^x, Λ_t^i) if (1.1) is attractive “in average” (see (2.6)). In what follows, we call (2.6) an “averaging condition”. As Example 2.3 below shows, our established theory, Theorem 2.2, covers more interesting models in contrast to existing results (see e.g. [28, Theorem 5.1]). By following the idea of argument for Theorem 2.2, Section 3 focus on existence and uniqueness of numerical invariant measure for RSDP (1.1) and (1.2) with additive noise and multiplicative noise respectively. In addition, we also reveal that numerical invariant

converges in the Wasserstein distance to the underlying one. For more details, please refer to Theorem 3.2. We point out that the Markov chain considered in Section 3 need not to be reversible. However, for the reversible case, by the principal eigenvalue approach (see e.g. Chen [7]), existence and uniqueness of numerical invariant measure can also be addressed if the principal eigenvalue is positive. This is elaborated in Section 4. Note that the Markov chain in Section 3 and 4 admits a finite state space. We proceed to the countable case in Section 5. For such case, a finite partition method due to Shao [22] and an M-matrix theory (see e.g. [15, Theorem 2.10, p.68]) are adopted to study existence of numerical invariant measure. More precisely, by a finite partition method, EM scheme with a countable state space is transformed into a new EM with a finite state space. Thus, the discrete-time semigroup generated by the EM scheme with a countable state space possesses an invariant measure provided that the one generated by the new EM with a finite state space does. Moreover, several examples (see Examples 2.3, 3.4, 4.2 and Remark 3.1) are constructed to demonstrate our theory established.

Throughout the paper, $c > 0$ is a generic constant which is independent of the time parameters and the stepsize, and may change from occurrence to occurrence.

2 Invariant Measure

To begin with, we introduce some notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. \mathcal{F}_0 contains all \mathbb{P} -null sets and $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s$). Let $\{\Lambda_t\}_{t \geq 0}$ be a continuous-time Markov chain with the state space $\mathbb{S} := \{1, 2, \dots, N\}$, $N < \infty$, and $\{W_t\}_{t \geq 0}$ an m -dimensional Brownian motion, independent of $\{\Lambda_t\}_{t \geq 0}$, defined on the probability space above. We assume that the Q -matrix $Q := (q_{ij})_{N \times N}$ is irreducible and conservative. So the Markov chain $\{\Lambda_t\}_{t \geq 0}$ has a unique stationary distribution $\mu := (\mu_1, \dots, \mu_N)$ which can be determined by solving linear equation

$$\mu Q = \mathbf{0}$$

subject to $\mu_1 + \dots + \mu_N = 1$ and $\mu_i > 0, i \in \mathbb{S}$. Here $\mathbf{0}$ is a zero vector. Let $\mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ stand for the family of all probability measures on $\mathbb{R}^n \times \mathbb{S}$. For $\xi = (\xi_1, \dots, \xi_n)^* \in \mathbb{R}^n$, $\xi \gg \mathbf{0}$ means each component $\xi_i > 0, i = 1, \dots, n$, in which A^* denotes the transpose of a vector or matrix A . For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$. For each $R > 0$, $B_R(0) := \{x \in \mathbb{R}^n : |x| \leq R\}$, the ball of radius R centered at 0. Let $\|A\|$ be the Hilbert-Schmidt norm of the matrix A . $\text{diag}(a_1, \dots, a_N)$ denotes the diagonal matrix whose diagonal entries starting in the upper left corner are a_1, \dots, a_N .

We assume that, in (1.1), $b : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ satisfy the local Lipschitz condition, i.e., for each $i \in \mathbb{S}$ and $R > 0$, there exists an $L_R > 0$ such that

$$|b(x, i) - b(y, i)| + \|\sigma(x, i) - \sigma(y, i)\| \leq L_R |x - y|, \quad x, y \in B_R(0). \quad (2.1)$$

Additionally, we assume that

(H) For each $i \in \mathbb{S}$ and $x, y \in \mathbb{R}^n$, there exist $c_0 > 0$ and $\beta_i \in \mathbb{R}$ such that

$$2\langle x, b(x, i) \rangle + \|\sigma(x, i)\|^2 \leq c_0 + \beta_i |x|^2, \quad (2.2)$$

and

$$2\langle x - y, b(x, i) - b(y, i) \rangle + \|\sigma(x, i) - \sigma(y, i)\|^2 \leq \beta_i |x - y|^2. \quad (2.3)$$

Remark 2.1. In (H), it is worth to pointing out that, for each $i \in \mathbb{S}$, β_i need not to be negative. On the other hand, without loss of generality, we assume that, for each $i \in \mathbb{S}$, (2.2) and (2.3) hold respectively with the same $\beta_i \in \mathbb{R}$ to avoid complex computation.

Under (2.1) and (H), (1.1) and (1.2) admit a unique non-explosive solution (X_t, Λ_t) , see e.g. [15, Theorem 3.17, p.93]. Throughout the paper, we write $(X_t^{x,i}, \Lambda_t^i)$ in lieu of (X_t, Λ_t) to highlight the initial data (x, i) . Define a metric $d(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{S}$ as below

$$d((x, i), (y, j)) := |x - y| + d_0(i, j),$$

where $d_0(i, j) = 0$ for $i = j$, otherwise $d_0(i, j) = 1$. Then $(\mathbb{R}^n \times \mathbb{S}, d(\cdot, \cdot), \mathcal{B}(\mathbb{R}^n \times \mathbb{S}))$ is a complete separable metric space. For two given probability measures μ and ν on $\mathbb{R}^n \times \mathbb{S}$, define

$$W_p(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^n \times \mathbb{S}} \int_{\mathbb{R}^n \times \mathbb{S}} d(x, y)^p \pi(dx, dy), \quad p \in (0, 1], \quad (2.4)$$

where $\mathcal{C}(\mu, \nu)$ denotes the set of all couplings of μ and ν . Let $P_t(x, i; dy \times \{j\})$ be the transition probability measure of the pair $(X_t^{x,i}, \Lambda_t^i)$, which is a time homogeneous Markov process (see e.g. [15, Theorem 3.28, p.105-106]). Recall that $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ is called an invariant measure of $(X_t^{x,i}, \Lambda_t^i)$ if

$$\pi(\Gamma \times \{i\}) = \sum_{j=1}^N \int_{\mathbb{R}^n} P_t(x, j; \Gamma \times \{i\}) \pi(dx \times \{j\}), \quad t \geq 0,$$

holds for any Borel set $\Gamma \in \mathcal{B}(\mathbb{R}^n)$ and $i \in \mathbb{S}$. For any $p > 0$, let

$$\text{diag}(\beta) := \text{diag}(\beta_1, \dots, \beta_N), \quad Q_p := Q + \frac{p}{2} \text{diag}(\beta), \quad \eta_p := - \max_{\gamma \in \text{spec}(Q_p)} \text{Re} \gamma, \quad (2.5)$$

where Q is the Q -matrix of $\{\Lambda_t\}_{t \geq 0}$, and $\text{spec}(Q_p)$ denotes the spectrum of Q_p .

The lemma below plays a crucial role for existence of an invariant measure of $(X_t^{x,i}, \Lambda_t^i)$.

Lemma 2.1. ([3, Proposition 4.2]) Let $N < \infty$ and assume that

$$\sum_{i=1}^N \mu_i \beta_i < 0. \quad (2.6)$$

Then, one has

- (i) $\eta_p > 0$ if $\max_{i \in \mathbb{S}} \beta_i \leq 0$;

(ii) $\eta_p > 0$ for $p < k$, where $k \in (0, \min_{i \in \mathbb{S}, \beta_i > 0} \{-2q_{ii}/\beta_i\})$ with $\max_{i \in \mathbb{S}} \beta_i > 0$.

Remark 2.2. The RSDP (1.1) and (1.2) is said to be attractive “in average” if (2.6) holds (see e.g. Bardet et al. [3]).

Our first main result in this paper is stated as below.

Theorem 2.2. Let $N < \infty$ and assume that (2.1), **(H)** and (2.6) hold. Then $(X_t^{x,i}, \Lambda_t^i)$ admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$.

Proof. Let $Q_{p,t} := e^{tQ_p}$, where Q_p is defined in (2.5). Then the spectral radius $\text{Ria}(Q_{p,t})$ of $Q_{p,t}$ equals to $e^{-\eta_p}$. Since all coefficients of $Q_{p,t}$ are positive, the Perron-Frobenius theorem (see e.g. [8, p.6]) yields that $-\eta_p$ is a simple eigenvalue of Q_p . Note that the eigenvalue of $Q_{p,t}$ corresponding to $e^{-\eta_p}$ is also an eigenvalue of Q_p corresponding to $-\eta_p$. The Perron-Frobenius theorem (see e.g. [8, p.6]) ensures that, for Q_p , there exists an eigenvector $\xi^{(p)} = (\xi_1^{(p)}, \dots, \xi_N^{(p)}) \gg \mathbf{0}$ corresponding to $-\eta_p$. Now, by Lemma 2.1 above, there exists some $p_0 > 0$ such that $\eta_p > 0$ for any $0 < p < p_0$. Hereinafter, fix a p with $0 < p < 1 \wedge p_0$ and the corresponding eigenvector $\xi^{(p)} \gg \mathbf{0}$. Then we obtain that

$$Q_p \xi^{(p)} = -\eta_p \xi^{(p)} \ll \mathbf{0}. \quad (2.7)$$

By the Itô formula and $p \in (0, 1 \wedge p_0)$, we obtain from **(H)** and (2.7) that

$$\begin{aligned} & e^{\eta_p t} \mathbb{E}((1 + |X_t^{x,i}|^2)^{p/2} \xi_{\Lambda_t^i}^{(p)}) \\ &= (1 + |x|^2)^{p/2} \xi_i^{(p)} + \mathbb{E} \int_0^t e^{\eta_p s} \{ \eta_p (1 + |X_s^{x,i}|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)} + (1 + |X_s^{x,i}|^2)^{p/2} (Q \xi^{(p)})(\Lambda_s^i) \\ & \quad + \frac{p}{2} (1 + |X_s^{x,i}|^2)^{(p-2)/2} (2 \langle X_s^{x,i}, b(X_s^{x,i}, \Lambda_s^i) \rangle + \|\sigma(X_s^{x,i}, \Lambda_s^i)\|^2) \xi_{\Lambda_s^i}^{(p)} \\ & \quad + \frac{p(p-2)}{2} (1 + |X_s^{x,i}|^2)^{(p-4)/2} |\sigma^*(X_s^{x,i}, \Lambda_s^i) X_s^{x,i}|^2 \xi_{\Lambda_s^i}^{(p)} \} ds \\ & \leq c(1 + |x|^p + e^{\eta_p t}) + \mathbb{E} \int_0^t e^{\rho s} \{ \eta_p \xi_{\Lambda_s^i}^{(p)} + (Q_p \xi^{(p)})(\Lambda_s^i) \} (1 + |X_s^{x,i}|^2)^{p/2} ds \\ & = c(1 + |x|^p + e^{\eta_p t}). \end{aligned}$$

This implies that

$$\sup_{t \geq 0} \mathbb{E} |X_t^{x,i}|^p < c \{ 1 + (1 + |x|^p) e^{-\eta_p t} \}. \quad (2.8)$$

Observe that $(X_t^{x,i}, \Lambda_t^i)$ is Feller continuous (see e.g. [31, Theorem 2.18, p.48]) and $B_R(0)$ is a compact subset of \mathbb{R}^n . For arbitrary $t > 0$, define a probability measure

$$\mu_t(\Gamma) := \frac{1}{t} \int_0^t P_s(x, i; \Gamma) ds, \quad \Gamma \in \mathcal{B}(\mathbb{R}^n \times \mathbb{S}).$$

Then, for any $\varepsilon > 0$, by (2.8) and Chebyshev's inequality, there exists $R > 0$ sufficiently large such that

$$\mu_t(B_R \times \mathbb{S}) = \frac{1}{t} \int_0^t P(s, x, i; B_R \times \mathbb{S}) ds \geq 1 - \frac{\sup_{t \geq 0} \mathbb{E} |X_t^{x,i}|^p}{R^p} \geq 1 - \varepsilon.$$

Hence $\{\mu_t\}_{t \geq 0}$ is tight and there exists an invariant measure of $(X_t^{x,i}, \Lambda_t^i)$ (see e.g. [6, Theorem 4.14, p.128]).

Next, we show uniqueness of invariant measure. Again, by the Itô formula, it follows from (2.7) and **(H)** that

$$\begin{aligned} e^{\eta_p t} \mathbb{E}(|X_t^{x,i} - X_t^{y,i}|^p \xi_{\Lambda_t^i}^{(p)}) &\leq |x - y|^p \xi_i^{(p)} + \mathbb{E} \int_0^t e^{\eta_p s} \{ \eta_p \xi_{\Lambda_s^i}^{(p)} + (Q_p \xi^{(p)})(\Lambda_s^i) \} |X_s^{x,i} - X_s^{y,i}|^p ds \\ &= |x - y|^p \xi_i^{(p)}, \end{aligned}$$

which gives

$$\mathbb{E}(|X_t^{x,i} - X_t^{y,i}|^p) \leq c|x - y|^p e^{-\eta_p t}. \quad (2.9)$$

Set

$$\tau := \inf\{t \geq 0 : \Lambda_t^i = \Lambda_t^j\}.$$

Since \mathbb{S} is a finite set, and Q is irreducible, there exists $\theta > 0$ such that

$$\mathbb{P}(\tau > t) \leq e^{-\theta t}, \quad t > 0. \quad (2.10)$$

Observe that (2.8) holds with different η_p and $\xi^{(p)}$ for any $0 < p < 1 \wedge p_0$. For $0 < p < 1 \wedge p_0$ above, choose $q > 1$ such that $0 < pq < 1 \wedge p_0$. By Hölder's inequality, we derive from (2.8)-(2.10) that

$$\begin{aligned} \mathbb{E}|X_t^{x,i} - X_t^{y,j}|^p &= \mathbb{E}(|X_t^{x,i} - X_t^{y,j}|^p \mathbf{1}_{\{\tau > t\}}) + \mathbb{E}(|X_t^{x,i} - X_t^{y,j}|^p \mathbf{1}_{\{\tau \leq t\}}) \\ &\leq (\mathbb{E}(|X_t^{x,i} - X_t^{y,j}|^{pq}))^{1/q} (\mathbb{P}(\tau > t))^{1-1/q} + \mathbb{E}(\mathbf{1}_{\{\tau \leq t\}} \mathbb{E}(|X_t^{x,i} - X_t^{y,j}|^p | \mathcal{F}_\tau)) \\ &\leq (\mathbb{E}(|X_t^{x,i} - X_t^{y,j}|^{pq}))^{1/q} (\mathbb{P}(\tau > t))^{1-1/q} + \mathbb{E}(\mathbf{1}_{\{\tau \leq t\}} \mathbb{E}(|X_{t-\tau}^{x,i, \Lambda_\tau^i} - X_{t-\tau}^{y,j, \Lambda_\tau^j}|^p)) \\ &\leq e^{-\frac{q-1}{2q}\theta t} (\mathbb{E}(|X_t^{x,i} - X_t^{y,j}|^{pq}))^{1/q} + c \mathbb{E}(\mathbf{1}_{\{\tau \leq t\}} e^{-\eta_p(t-\tau)} \mathbb{E}|X_{t-\tau}^{x,i, \Lambda_\tau^i} - X_{t-\tau}^{y,j, \Lambda_\tau^j}|^p) \\ &\leq ce^{-\rho t}, \end{aligned}$$

where $\rho := \frac{(q-1)\theta}{2q} \wedge \frac{\eta_p}{2}$. Note that

$$W_p(\delta_{(x,i)} P_t, \delta_{(y,j)} P_t) \leq \mathbb{E}|X_t^{x,i} - X_t^{y,j}|^p + \mathbb{P}(\Lambda_t^i \neq \Lambda_t^j). \quad (2.11)$$

Assume that $\pi, \nu \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ are invariant measures of $(X_t^{x,i}, \Lambda_t^i)$. By the Kantorovich-Rubinstein duality formula (see e.g. [25, Theorem 5.10]), it follows from (2.11) that

$$\begin{aligned} W_p(\pi, \nu) &= W_p(\pi P_t, \nu P_t) = \sup_{\varphi: \text{Lip}(\varphi)=1} \left\{ \int_{\mathbb{R}^n \times \mathbb{S}} \varphi(x, i) d(\pi P_t) - \int_{\mathbb{R}^n \times \mathbb{S}} \varphi(x, i) d(\nu P_t) \right\} \\ &\leq \int_{\mathbb{R}^n \times \mathbb{S}} \int_{\mathbb{R}^n \times \mathbb{S}} \pi(dx \times \{i\}) \nu(dy \times \{j\}) W_p(\delta_{(x,i)} P_t, \delta_{(y,j)} P_t) \\ &\longrightarrow 0, \end{aligned} \quad (2.12)$$

where

$$\text{Lip}(\varphi) := \sup \left\{ \frac{\varphi(x, i) - \varphi(y, j)}{d^p((x, i), (y, j))} : (x, i) \neq (y, j) \right\}.$$

Hence, uniqueness of invariant measure follows. \square

Next, we provide an example to demonstrate that our theory is more general than that of the existing literature.

Example 2.3. Let $\{\Lambda_t\}_{t \geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0, 1\}$ with the generator

$$Q = \begin{pmatrix} -4 & 4 \\ \gamma & -\gamma \end{pmatrix} \quad (2.13)$$

with some $\gamma > 0$. Consider a scalar Ornstein-Uhlenback (O-U) process with regime switching

$$dX_t = \alpha_{\Lambda_t} X_t dt + \sigma_{\Lambda_t} dW_t, \quad t > 0, \quad X_0 = x, \quad \Lambda_0 = i_0 \in \mathbb{S}, \quad (2.14)$$

where $\alpha, \sigma : \mathbb{S} \mapsto \mathbb{R}$ such that $\alpha_0 = 1$, and $\alpha_1 = -1/2$.

By an M-Matrix approach, $(X_t^{x,i}, \Lambda_t^i)$, determined by (2.13) and (2.14), has a unique invariant measure for $\gamma \in (0, 1)$ (see e.g. [28, Example 5.1]). It is easy to see that the stationary distribution of $\{\Lambda_t\}_{t \geq 0}$ is

$$\mu = (\mu_0, \mu_1) = \left(\frac{\gamma}{4 + \gamma}, \frac{4}{4 + \gamma} \right).$$

For the O-U process (2.14), $\beta_0 = 2$, $\beta_1 = -1$ in **(H)** and (2.6) holds with $\gamma \in (0, 2)$. So, by Theorem 2.2, $(X_t^{x,i}, \Lambda_t^i)$, determined by (2.13) and (2.14), admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R} \times \mathbb{S})$ for $\gamma \in (0, 2)$. This means that our result cannot be covered by the existing results.

Remark 2.3. In (1.1), let $b, \sigma : \mathbb{R} \times \mathbb{S} \mapsto \mathbb{R}$, with $b(0, i) = \sigma(0, i) \equiv 0$ for $i \in \mathbb{S}$, satisfy the global Lipschitz condition, and $\{W_t\}_{t \geq 0}$ be a scalar Brownian motion. For each $i \in \mathbb{S}$, assume that there exists $\beta_i \in \mathbb{R}$ such that

$$2(x - y)(b(x, i) - b(y, i)) \leq \beta_i |x - y|^2, \quad x, y \in \mathbb{R}.$$

By [13, Lemma 3.2, p.120], for $X_0^{x,i} = x \neq 0$, $\mathbb{P}(X_t^{x,i} \neq 0 \text{ on } t \geq 0) = 1$, i.e., almost all the sample path of any solution starting from a non-zero state will never reach the origin. In the sequel, without loss of generality, we assume $X_0^{x,i} = x > 0$. Following the first part of argument for Theorem 2.2, and applying Itô's formula to $\mathbb{E}((X_t^{x,i})^p \xi_{\Lambda_t^i}^{(p)})$, we can deduce that $(X_t^{x,i}, \Lambda_t^i)$ admits an invariant measure $\pi \in \mathcal{P}(\mathbb{R} \times \mathbb{S})$ whenever $\sum_{i=1}^N \mu_i \beta_i < 0$. Next, for any $x > y$, by the comparison theorem (see e.g. [11, Theorem 1.1, p.352]), we have $X_t^{x,i} > X_t^{y,i}$ a.s. Then, by imitating the second part of argument for Theorem 2.2 and utilizing Itô's formula to $\mathbb{E}((X_t^{x,i} - X_t^{y,i})^p \xi_{\Lambda_t^i}^{(p)})$, uniqueness of invariant measure for $(X_t^{x,i}, \Lambda_t^i)$ follows provided that $\sum_{i=1}^N \mu_i \beta_i < 0$. Therefore, for a scalar RSDP, existence and uniqueness of invariant measure can be determined only by the drift coefficient in some cases.

3 Numerical Invariant Measure

In the last section, under the “averaging condition” (2.6), we discuss existence and uniqueness of invariant measure for the semigroup generated by the pair $(X_t^{x,i}, \Lambda_t^i)$, determined by (1.1) and (1.2). In this section, assuming $N < \infty$ we turn to study existence and uniqueness of invariant measure for the semigroup generated by the EM scheme constructed as below. For a given stepsize $\delta \in (0, 1)$, we define the discrete-time EM scheme associated with (1.1) as follows

$$\bar{Y}_{(k+1)\delta}^{x,i} := \bar{Y}_{k\delta}^{x,i} + b(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)\delta + \sigma(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)\Delta W_k, \quad k \geq 0, \quad \bar{Y}_0^{x,i} = x, \Lambda_0^i = i \in \mathbb{S}, \quad (3.1)$$

where $\Delta W_k := W_{(k+1)\delta} - W_{k\delta}$ stands for the Brownian motion increment. For convenience, we also need the following continuous-time EM scheme

$$Y_t^{x,i} := x + \int_0^t b(\bar{Y}_{[s/\delta]\delta}^{x,i}, \Lambda_{[s/\delta]\delta}^i)ds + \int_0^t \sigma(\bar{Y}_{[s/\delta]\delta}^{x,i}, \Lambda_{[s/\delta]\delta}^i)dW_s, \quad t \geq 0, \quad \Lambda_0^i = i \in \mathbb{S}, \quad (3.2)$$

where, for $a \geq 0$, $[a]$ denotes the integer part of a . Note that $Y_{k\delta}^{x,i} = \bar{Y}_{k\delta}^{x,i}$, $k \geq 0$. That is, the discrete-time EM scheme (3.1) coincides with the continuous-time EM scheme (3.2) at the gridpoints whenever they enjoy the same starting points. Hence, for some quantitative analysis, it is sufficient to focus on $\{Y_t^{x,i}\}_{t \geq 0}$ instead of $\{\bar{Y}_{k\delta}^{x,i}\}_{k \geq 0}$.

Let $P_{k\delta}^\delta(x, i; dy \times \{j\})$ be the transition probability kernel of $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$, which is a time homogeneous Markov chain (see e.g. [15, Theorem 6.14, p.250]). If $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ satisfies

$$\pi^\delta(\Gamma \times \{i\}) = \sum_{j=1}^N \int_{\mathbb{R}^n} P_{k\delta}^\delta(x, j; \Gamma \times \{i\}) \pi^\delta(dx \times \{j\}), \quad \Gamma \in \mathcal{B}(\mathbb{R}^n),$$

then we call $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ an invariant measure of $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$. Moreover, the invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ is also said to be a numerical invariant measure of $(X_t^{x,i}, \Lambda_t^i)$.

In this section, we further assume that $b : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ are globally Lipschitzian, i.e., for each $i \in \mathbb{S}$ and $x, y \in \mathbb{R}^n$, there exists an $L > 0$ such that

$$|b(x, i) - b(y, i)| + \|\sigma(x, i) - \sigma(y, i)\| \leq L|x - y|. \quad (3.3)$$

This implies the linear growth condition:

$$|b(x, i)| + \|\sigma(x, i)\| \leq L_0 + L|x|, \quad x \in \mathbb{R}^n, \quad (3.4)$$

where $L_0 := \max_{i \in \mathbb{S}} \{|b(0, i)| + \|\sigma(0, i)\|\}$.

In the sequel, we shall investigate existence and uniqueness of invariant measure for $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$, determined by (3.1) and (1.2) with additive noise and multiplicative noise case respectively.

3.1 Additive Noise Case

We here consider (1.1) with additive noise in the form

$$dX_t = b(X_t, \Lambda_t)dt + \sigma(\Lambda_t)dW_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^n, \Lambda_0 = i \in \mathbb{S}, \quad (3.5)$$

where $\sigma : \mathbb{S} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$, and the other quantities are defined exactly as in (1.1) and (1.2). Moreover, the EM scheme $\bar{Y}_{k\delta}^{x,i}$ associated with (3.5) is constructed as in (3.1) with $\sigma(\cdot, \cdot) \equiv \sigma(\cdot)$. In what follows, $\xi^{(p)} \gg \mathbf{0}$ is the eigenvector Q_p , defined in (2.5), with the corresponding eigenvalue $-\eta_p < 0$ for $0 < p < 1 \wedge p_0$, i.e., (2.7) holds. Let

$$\hat{\xi}_0 := \max_{i \in \mathbb{S}} \xi_i^{(p)}, \quad \bar{\xi}_0 := (\min_{i \in \mathbb{S}} \xi_i^{(p)})^{-1}, \quad \beta_0 := \max_{i \in \mathbb{S}} |\beta_i|, \quad q_0 := \max_{i \in \mathbb{S}} (-q_{ii}). \quad (3.6)$$

Set

$$\alpha := p\{\beta_0 + 4L^2(3 + 4\beta_0) + 4^{(2+p)/2}\beta_0(4^{p/2}L^p + q_0\hat{\xi}_0\bar{\xi}_0)\}. \quad (3.7)$$

Our main result in this subsection is as follows.

Theorem 3.1. Let $N < \infty$, and assume further that **(H)**, (2.6), and (3.3) hold. Then, for

$$\delta < (1/(16L^2)) \wedge (\eta_p/\alpha)^{2/p}, \quad (3.8)$$

$(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ admits a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$.

Proof. We divide the whole proof into two parts.

(i) Existence of an Invariant Measure. For each integer $q \geq 1$, define the measure

$$\mu_q(B_R(0) \times \mathbb{S}) := \frac{1}{q} \sum_{k=0}^q \mathbb{P}((\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i) \in B_R(0) \times \mathbb{S}).$$

To show existence of an invariant measure, it suffices to show that, for any $(x, i) \in \mathbb{R}^n \times \mathbb{S}$,

$$\sup_{k \geq 0} \mathbb{E}|\bar{Y}_{k\delta}^{x,i}|^p < \infty, \quad p \in (0, 1 \wedge p_0). \quad (3.9)$$

Indeed, if so, the Chebyshev inequality yields that the measure sequence $\{\mu_q(\cdot)\}_{q \geq 1}$ is tight. Then, one can extract a subsequence which converges weakly to an invariant measure (see e.g. Meyn and Tweedie [17]).

In what follows, we prove that (3.9) holds. Let $W_{t,\delta} := |W_t - W_{\lfloor t/\delta \rfloor \delta}|^2$. By (3.4) and (3.8), one has

$$|\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i}|^2 \leq c(\delta + W_{t,\delta}) + 4|Y_t^{x,i}|^2, \quad (3.10)$$

and

$$|Y_t^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i}|^2 \leq c(\delta + W_{t,\delta}) + 16L^2\delta|Y_t^{x,i}|^2. \quad (3.11)$$

By applying Itô's formula, for any $\rho > 0$ and $p \in (0, 1 \wedge p_0)$, it follows from (2.2) and (2.7) that

$$\begin{aligned}
& e^{\rho t} \mathbb{E}((1 + |Y_t^{x,i}|^2)^{p/2} \xi_{\Lambda_t^i}^{(p)}) \\
& \leq (1 + |x|^2)^{p/2} \xi_i^{(p)} + \mathbb{E} \int_0^t e^{\rho s} \{ (\rho \xi_{\Lambda_s^i}^{(p)} + (Q\xi^{(p)})(\Lambda_s^i)) (1 + |Y_s^{x,i}|^2)^{p/2} \\
& \quad + \frac{p}{2} (1 + |Y_s^{x,i}|^2)^{(p-2)/2} (2 \langle Y_s^{x,i}, b(\bar{Y}_{[s/\delta]\delta}^{x,i}, \Lambda_{[s/\delta]\delta}^i) \rangle + \|\sigma\|^2) \xi_{\Lambda_s^i}^{(p)} \} ds \\
& \leq (1 + |x|^2)^{p/2} \xi_i^{(p)} + c e^{\rho t} + \mathbb{E} \int_0^t e^{\rho s} \{ \rho \xi_{\Lambda_s^i}^{(p)} + (Q\xi^{(p)})(\Lambda_s^i) + \frac{p}{2} \beta_{\Lambda_s^i} \xi_{\Lambda_s^i}^{(p)} \} (1 + |Y_s^{x,i}|^2)^{p/2} ds \\
& \quad + \mathbb{E} \int_0^t e^{\rho s} (1 + |Y_s^{x,i}|^2)^{(p-2)/2} \{ \Theta_1(s) + \Theta_2(s) \} ds \\
& \leq c(1 + |x|^p + e^{\rho t}) - (\eta_p - \rho) \mathbb{E} \int_0^t e^{\rho s} (1 + |Y_s^{x,i}|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)} ds \\
& \quad + \mathbb{E} \int_0^t e^{\rho s} (1 + |Y_s^{x,i}|^2)^{(p-2)/2} \{ \Theta_1(s) + \Theta_2(s) \} ds,
\end{aligned}$$

where

$$\begin{aligned}
\Theta_1(t) &:= p \langle Y_t^{x,i} - \bar{Y}_{[t/\delta]\delta}^{x,i}, b(\bar{Y}_{[t/\delta]\delta}^{x,i}, \Lambda_{[t/\delta]\delta}^i) \rangle \xi_{\Lambda_t^i}^{(p)} + \frac{p}{2} \beta_{\Lambda_t^i} |\bar{Y}_{[t/\delta]\delta}^{x,i} - Y_t^{x,i}|^2 \xi_{\Lambda_t^i}^{(p)} \\
&\quad + p \beta_{\Lambda_t^i} \langle Y_t^{x,i}, \bar{Y}_{[t/\delta]\delta}^{x,i} - Y_t^{x,i} \rangle \xi_{\Lambda_t^i}^{(p)},
\end{aligned}$$

and

$$\Theta_2(t) := \frac{p}{2} (\beta_{\Lambda_{[t/\delta]\delta}^i} - \beta_{\Lambda_t^i}) |\bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \xi_{\Lambda_t^i}^{(p)}.$$

By the fundamental inequality: $a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b$ with $a, b > 0$ and $\nu \in (0, 1)$, (3.4), (3.10) and (3.11) yield that

$$\begin{aligned}
\Theta_1(t) &\leq \frac{p}{2} \{ (\beta_0 + (1 + \beta_0)(\sqrt{\delta})^{-1}) |Y_t^{x,i} - \bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \xi_{\Lambda_t^i}^{(p)} \\
&\quad + \frac{p\beta_0}{2} \sqrt{\delta} |Y_t^{x,i}|^2 \xi_{\Lambda_t^i}^{(p)} + \frac{p}{2} \sqrt{\delta} |b(\bar{Y}_{[t/\delta]\delta}^{x,i}, \Lambda_{[t/\delta]\delta}^i)|^2 \xi_{\Lambda_t^i}^{(p)} \\
&\leq c(1 + W_{t,\delta}) + c \{ (\beta_0 + (1 + \beta_0)(\sqrt{\delta})^{-1}) \} (\delta + W_{t,\delta}) \xi_{\Lambda_t^i}^{(p)} \\
&\quad + p \{ \beta_0 + 4L^2(3 + 4\beta_0) \} \sqrt{\delta} |Y_t^{x,i}|^2 \xi_{\Lambda_t^i}^{(p)}.
\end{aligned} \tag{3.12}$$

For any $t \leq \delta$, due to $q_{ii} < 0$ one has

$$\mathbb{P}(\Lambda_t^i \neq \Lambda_0^i = i) = 1 - \mathbb{P}(\Lambda_t^i = \Lambda_0^i) \leq 1 - e^{q_{ii}t} \leq 1 - e^{q_{ii}\delta} \leq -q_{ii}\delta \leq q_0\delta.$$

This further gives that

$$\begin{aligned}
\mathbb{E}(|Y_{[t/\delta]\delta}^{x,i}|^p \mathbf{1}_{\Lambda_{[t/\delta]\delta}^i \neq \Lambda_t^i}) &= \mathbb{E}(\mathbb{E}(|Y_{[t/\delta]\delta}^{x,i}|^p \mathbf{1}_{\Lambda_t^i \neq \Lambda_{[t/\delta]\delta}^i} | \mathcal{F}_{[t/\delta]\delta})) \\
&= \mathbb{E}(|Y_{[t/\delta]\delta}^{x,i}|^p \mathbb{E}(\mathbf{1}_{\Lambda_t^i \neq \Lambda_{[t/\delta]\delta}^i} | \mathcal{F}_{[t/\delta]\delta})) \\
&= \mathbb{E}(|Y_{[t/\delta]\delta}^{x,i}|^p \mathbb{E}(\mathbf{1}_{\Lambda_t^i \neq \Lambda_{[t/\delta]\delta}^i} | \Lambda_{[t/\delta]\delta}^i)) \\
&\leq q_0 \delta \mathbb{E}|Y_{[t/\delta]\delta}^{x,i}|^p,
\end{aligned} \tag{3.13}$$

where we have used that $\{W_t\}_{t \geq 0}$ is independent of $\{\Lambda_t\}_{t \geq 0}$. Furthermore, in light of (3.10), (3.11) and (3.13), it follows that

$$\begin{aligned}
& \mathbb{E} \int_0^t e^{\rho s} (1 + |Y_s^{x,i}|^2)^{(p-2)/2} \Theta_2(s) ds \\
& \leq p\beta_0 \mathbb{E} \int_0^t e^{\rho s} (1 + |Y_s^{x,i}|^2)^{(p-2)/2} |\bar{Y}_{\lfloor s/\delta \rfloor \delta}^{x,i}|^2 \mathbf{1}_{\Lambda_s^i \neq \Lambda_{\lfloor s/\delta \rfloor \delta}^i} \xi_{\Lambda_s^i}^{(p)} ds \\
& \leq ce^{\rho t} + 4p\beta_0 \widehat{\xi}_0 \mathbb{E} \int_0^t e^{\rho s} (1 + |\bar{Y}_{\lfloor s/\delta \rfloor \delta}^{x,i}|^2)^{p/2} \mathbf{1}_{\Lambda_s^i \neq \Lambda_{\lfloor s/\delta \rfloor \delta}^i} ds \\
& \quad + 4p\beta_0 \mathbb{E} \int_0^t e^{\rho s} (1 + |Y_s^{x,i} - \bar{Y}_{\lfloor s/\delta \rfloor \delta}^{x,i}|^2)^{p/2} \mathbf{1}_{\Lambda_s^i \neq \Lambda_{\lfloor s/\delta \rfloor \delta}^i} \xi_{\Lambda_s^i}^{(p)} ds \\
& \leq ce^{\rho t} + 4^{(2+p)/2} p\beta_0 (4^{p/2} L^p + q_0 \widehat{\xi}_0 \bar{\xi}_0) \delta^{p/2} \int_0^t e^{\rho s} \mathbb{E}((1 + |Y_s^{x,i}|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)}) ds.
\end{aligned} \tag{3.14}$$

Consequently, according to (3.12) and (3.14), we arrive at

$$e^{\rho t} \mathbb{E}((1 + |Y_t^{x,i}|^2)^{p/2} \xi_{\Lambda_t^i}^{(p)}) \leq c(1 + |x|^p + e^{\rho t}) - (\eta_p - \rho - \alpha \delta^{p/2}) \int_0^t e^{\rho s} \mathbb{E}((1 + |Y_s|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)}) ds,$$

where $\alpha > 0$ is defined in (3.7). Taking $\rho = \eta_p - \alpha \delta^{p/2} > 0$ due to (3.8) leads to (3.9).

(ii) Uniqueness of Invariant Measure. By checking the second part of argument for Theorem 3.3, we need only to show that

$$\mathbb{E} |\bar{Y}_{k\delta}^{x,i} - \bar{Y}_{k\delta}^{y,i}|^p \leq c|x - y|^p e^{-\eta_p t}, \quad x, y \in \mathbb{R}^n. \tag{3.15}$$

For $\delta \in (0, 1)$ such that (3.8) holds, note that

$$|\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i}|^2 \leq 4|Y_t^{x,i} - Y_t^{y,i}|^2, \tag{3.16}$$

and

$$|Y_t^{x,i} - Y_t^{y,i} - (\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i})|^2 \leq 4L^2 \delta |Y_t^{x,i} - Y_t^{y,i}|^2. \tag{3.17}$$

For arbitrary $\varepsilon > 0$, $\rho > 0$, and $p \in (0, 1 \wedge p_0)$, by the Itô formula and (H), it follows from (2.7) that

$$\begin{aligned}
& \mathbb{E}(e^{\rho t} (\varepsilon + |Y_t^{x,i} - Y_t^{y,i}|^2)^{p/2} \xi_{\Lambda_t^i}^{(p)}) \\
& \leq c(\varepsilon^{p/2} + |x - y|^p) + \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{(p-2)/2} \{(\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)(\rho \xi_{\Lambda_s^i}^{(p)} + (Q\xi^{(p)})(\Lambda_s^i)) \\
& \quad + p(Y_s^{x,i} - Y_s^{y,i}, b(\bar{Y}_{\lfloor s/\delta \rfloor \delta}^{x,i}, \Lambda_{\lfloor s/\delta \rfloor \delta}^i) - b(Y_{\lfloor s/\delta \rfloor \delta}^{y,i}, \Lambda_{\lfloor s/\delta \rfloor \delta}^i)) \xi_{\Lambda_s^i}^{(p)}\} ds \\
& \leq c(\varepsilon^{p/2} + |x - y|^p) + \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{p/2} \{\rho \xi_{\Lambda_s^i}^{(p)} + (Q\xi^{(p)})(\Lambda_s^i) + \frac{p}{2} \beta_{\Lambda_s^i} \xi_{\Lambda_s^i}^{(p)}\} ds \\
& \quad + \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{(p-2)/2} \{\Upsilon_1(s) + \Upsilon_2(s)\} ds \\
& \leq c(\varepsilon^{p/2} + |x - y|^p) - (\eta_p - \rho) \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)} ds
\end{aligned}$$

$$+ \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{(p-2)/2} \{\Upsilon_1(s) + \Upsilon_2(s)\} ds,$$

where

$$\begin{aligned} \Upsilon_1(t) &:= p \langle Y_t^{x,i} - Y_t^{y,i} - (\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i}), b(\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i}, \Lambda_{\lfloor t/\delta \rfloor \delta}^i) - b(\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i}, \Lambda_{\lfloor t/\delta \rfloor \delta}^i) \rangle \xi_{\Lambda_t^i}^{(p)}, \\ &\quad + \frac{p}{2} \beta_{\Lambda_t^i} |\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i} - (Y_t^{x,i} - Y_t^{y,i})|^2 \xi_{\Lambda_t^i}^{(p)}, \\ &\quad + p \beta_{\Lambda_t^i} \langle Y_t^{x,i} - Y_t^{y,i}, \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i} - (Y_t^{x,i} - Y_t^{y,i}) \rangle \xi_{\Lambda_t^i}^{(p)} \\ \Upsilon_2(t) &:= \frac{p}{2} (\beta_{\Lambda_{\lfloor t/\delta \rfloor \delta}^i} - \beta_{\Lambda_t^i}) |\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i}|^2 \xi_{\Lambda_t^i}^{(p)}. \end{aligned}$$

Observe from (3.3), (3.16) and (3.17) that

$$\begin{aligned} \Upsilon_1(t) &\leq \frac{p}{2} \{\beta_0 + (1 + \beta_0)(\sqrt{\delta})^{-1}\} |Y_t^{x,i} - Y_t^{y,i} - (\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i})|^2 \xi_{\Lambda_t^i}^{(p)} \\ &\quad + \frac{pL^2}{2} \sqrt{\delta} |\bar{Y}_{\lfloor t/\delta \rfloor \delta}^{x,i} - \bar{Y}_{\lfloor t/\delta \rfloor \delta}^{y,i}|^2 \xi_{\Lambda_t^i}^{(p)} + \frac{p\beta_0}{2} \sqrt{\delta} |Y_t^{x,i} - Y_t^{y,i}|^2 \xi_{\Lambda_t^i}^{(p)} \\ &\leq p\{4(1 + \beta_0)L^2 + \beta_0\} \sqrt{\delta} |Y_t^{x,i} - Y_t^{y,i}|^2 \xi_{\Lambda_t^i}^{(p)}. \end{aligned}$$

As (3.14) was done, by virtue of (3.3), (3.16), and (3.17), we deduce that

$$\begin{aligned} &\mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{(p-2)/2} \Upsilon_2(s) ds \\ &\leq c\varepsilon^{p/2} + 4^{(2+p)/2} p \beta_0 (q_0 \widehat{\xi}_0 \bar{\xi}_0 + L^p) \delta^{p/2} \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)} ds. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &\mathbb{E}(e^{\rho t} (\varepsilon + |Y_t^{x,i} - Y_t^{y,i}|^2)^{p/2} \xi_{\Lambda_t^i}^{(p)}) \\ &\leq c\varepsilon^{p/2} - (\eta_p - \rho - \alpha \delta^{p/2}) \mathbb{E} \int_0^t e^{\rho s} (\varepsilon + |Y_s^{x,i} - Y_s^{y,i}|^2)^{p/2} \xi_{\Lambda_s^i}^{(p)} ds, \end{aligned}$$

where $\alpha > 0$ is defined as in (3.7). Then (3.15) follows by choosing $\rho = \eta_p - \alpha \delta^{p/2} > 0$ own to (3.8) and taking $\varepsilon \downarrow 0$. \square

Remark 3.1. Let us reexamine the Example 2.3. Note that **(H)**, (2.6), and (3.3) hold with $\beta_1 = 1$, $\beta_2 = -\frac{1}{2}$, $\gamma \in (0, 2)$, and $L = 1$ respectively. So, $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$, associated with (2.14) and (2.13), admits a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R} \times \mathbb{S})$ whenever the stepsize $\delta < (1/16) \wedge (\eta_p/\alpha)^{2/p}$ for $\alpha = p\{29 + 4^{(2+p)/2}(4^{p/2} + 4\widehat{\xi}_0\bar{\xi}_0)\}$.

The following theorem reveals that numerical invariant measure π^δ converges in the Wasserstein distance to the underlying one.

Theorem 3.2. Under the assumptions of Theorem 3.1, there exists $c > 0$ such that

$$W_p(\pi, \pi^\delta) \leq c\delta^{p/2}, \quad p < 1 \wedge p_0,$$

where $p_0 > 0$ is introduced in the argument of Theorem 3.3.

Proof. For any $p < 1 \wedge p_0$, note that

$$W_p(\delta_{(x,i)}P_{k\delta}, \pi) \leq \int_{\mathbb{R}^n \times \mathbb{S}} \pi(dy \times \{j\}) W_p(\delta_{(x,i)}P_{k\delta}, \delta_{(y,j)}P_{k\delta}),$$

and

$$W_p(\delta_{(x,i)}P_{k\delta}^\delta, \pi^\delta) \leq \int_{\mathbb{R}^n \times \mathbb{S}} \pi^\delta(dy \times \{j\}) W_p(\delta_{(x,i)}P_{k\delta}^\delta, \delta_{(y,j)}P_{k\delta}^\delta).$$

Then, by a close inspection of arguments for Theorem 2.2 and 3.3, for $\delta \in (0, 1)$ such that (3.8), there exist $k > 0$ sufficiently large and $c_1 > 0$ such that

$$W_p(\delta_{(x,i)}P_{k\delta}, \pi) + W_p(\delta_{(x,i)}P_{k\delta}^\delta, \pi^\delta) \leq c_1\delta^{p/2}.$$

Moreover, for fixed $k > 0$ above, it follows from [30, Theorem 3.1] that

$$W_p(\delta_{(x,i)}P_{k\delta}, \delta_{(x,i)}P_{k\delta}^\delta) \leq c_2\delta^{p/2}$$

for some $c_2 > 0$. Then the desired assertion follows from the triangle inequality. \square

3.2 Multiplicative Noise Case

In the previous subsection, we discuss existence and uniqueness of numerical invariant measures for the RSDP (1.1) and (1.2) with additive noise. While, in this subsection, we turn to study the case of multiplicative noise. We further assume that

$$\min_{i \in \mathbb{S}, \beta_i > 0} \{-q_{ii}/\beta_i\} > 1. \quad (3.18)$$

Under this condition, by Lemma 2.1 (ii) we can take $p = 2$ in (2.5). Set

$$\text{diag}(\beta) := \text{diag}(\beta_1, \dots, \beta_N), \quad Q_2 := Q + \text{diag}(\beta), \quad \eta_2 := -\max_{\gamma \in \text{spec}(Q_2)} \text{Re}\gamma,$$

where Q is the Q -matrix of $\{\Lambda_t\}_{t \geq 0}$, and $\text{spec}(Q_2)$ denotes the spectrum of Q_2 . Following an argument of (2.7), we can deduce from (2.6) and (3.18) that there exists an eigenvector $\xi^{(2)} \gg \mathbf{0}$ of Q_2 with eigenvalue $-\eta_2 < 0$ such that

$$Q_2 \xi^{(2)} = -\eta_2 \xi^{(2)} \ll \mathbf{0}. \quad (3.19)$$

Set

$$\widehat{\xi}_2 := \max_{i \in \mathbb{S}} \xi_i^{(2)}, \quad \bar{\xi}_2 := \min_{i \in \mathbb{S}} (\xi_i^{(2)})^{-1}, \quad \beta := \{(1 + 12q_0)\beta_0 + 8L^2(5 + 6\beta_0)\} \widehat{\xi}_2 \bar{\xi}_2, \quad (3.20)$$

where $q_0, \beta_0 > 0$ are defined as in (3.6).

Our main result in this subsection is presented as follows.

Theorem 3.3. Let $N < \infty$, and assume further that **(H)**, (2.6), (3.3), and (3.18) hold. For

$$\delta < (1/(32L^2)) \wedge (\eta_2/\beta)^2, \quad (3.21)$$

where $\beta > 0$ is given in (3.20), $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ admits a unique measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$.

Proof. The ideas of argument for Theorem 3.3 is analogous to that of Theorem 3.1. However, we herein give an outline of the argument to point out some corresponding differences.

(i) Existence of an Invariant Measure. To end this, it is sufficient to show that

$$\sup_{k \geq 0} \mathbb{E} |\bar{Y}_{k\delta}^{x,i}|^2 < \infty, \quad (x, i) \in \mathbb{R}^n \times \mathbb{S}. \quad (3.22)$$

From (3.2) and (3.4), one has

$$\mathbb{E} |Y_t^{x,i} - \bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \leq c\delta + 8L^2\delta \mathbb{E} |Y_{[t/\delta]\delta}^{x,i}|^2.$$

This further leads to

$$\mathbb{E} |\bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \leq 2\mathbb{E} |Y_t^{x,i}|^2 + 2\mathbb{E} |Y_t^{x,i} - \bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \leq 2\mathbb{E} |Y_t^{x,i}|^2 + 2(c + 8L^2\mathbb{E} |Y_{[t/\delta]\delta}^{x,i}|^2)\delta.$$

Hence, due to (3.21),

$$\mathbb{E} |\bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \leq c + 4\mathbb{E} |Y_t^{x,i}|^2 \quad \text{and} \quad \mathbb{E} |Y_t^{x,i} - \bar{Y}_{[t/\delta]\delta}^{x,i}|^2 \leq c\delta + 32L^2\delta \mathbb{E} |Y_t^{x,i}|^2. \quad (3.23)$$

By Itô's formula, for any $\rho > 0$, it follows from (2.2) and (3.19) that

$$\begin{aligned} & e^{\rho t} \mathbb{E} (|Y_t^{x,i}|^2 \xi_{\Lambda_t^i}^{(2)}) \\ &= |x|^2 \xi_i^{(2)} + \mathbb{E} \int_0^t e^{\rho s} \{ \rho |Y_s^{x,i}|^2 \xi_{\Lambda_s^i}^{(2)} + |Y_s^{x,i}|^2 (Q\xi^{(2)})(\Lambda_s^i) \\ &\quad + (2\langle Y_s^{x,i}, b(\bar{Y}_{[s/\delta]\delta}^{x,i}, \Lambda_{[s/\delta]\delta}^i) \rangle + \|\sigma(\bar{Y}_{[s/\delta]\delta}^{x,i}, \Lambda_{[s/\delta]\delta}^i)\|^2) \xi_{\Lambda_s^i}^{(2)} \} ds \\ &\leq c(|x|^2 + e^{\rho t}) + \mathbb{E} \int_0^t e^{\rho s} \{ \rho \xi_{\Lambda_s^i}^{(2)} + (Q\xi^{(2)})(\Lambda_s^i) + \beta_{\Lambda_s^i} \xi_{\Lambda_s^i}^{(2)} \} |Y_s^{x,i}|^2 ds \\ &\quad + \int_0^t e^{\rho s} \{ \Lambda_1(s) + \Lambda_2(s) \} ds \\ &\leq c(|x|^2 + e^{\rho t}) - \int_0^t e^{\rho s} (\eta_2 - \rho) \mathbb{E} (|Y_s^{x,i}|^2 \xi_{\Lambda_s^i}^{(2)}) ds + \int_0^t e^{\rho s} \{ \Lambda_1(s) + \Lambda_2(s) \} ds, \end{aligned} \quad (3.24)$$

where

$$\begin{aligned} \Lambda_1(t) &:= 2\mathbb{E} (\langle Y_t^{x,i} - \bar{Y}_{[t/\delta]\delta}^{x,i}, b(\bar{Y}_{[t/\delta]\delta}^{x,i}, \Lambda_{[t/\delta]\delta}^i) \rangle \xi_{\Lambda_t^i}^{(2)}) + \mathbb{E} (\beta_{\Lambda_t^i} |Y_{[t/\delta]\delta}^{x,i} - Y_t^{x,i}|^2 \xi_{\Lambda_t^i}^{(2)}) \\ &\quad + 2\mathbb{E} (\beta_{\Lambda_t^i} \langle Y_t^{x,i}, Y_{[t/\delta]\delta}^{x,i} - Y_t^{x,i} \rangle \xi_{\Lambda_t^i}^{(2)}), \\ \Lambda_2(t) &:= \mathbb{E} ((\beta_{\Lambda_{[t/\delta]\delta}^i} - \beta_{\Lambda_t^i}) |Y_{[t/\delta]\delta}^{x,i}|^2 \xi_{\Lambda_t^i}^{(2)}). \end{aligned} \quad (3.25)$$

From (3.4) and (3.23), we derive that

$$\begin{aligned}\Lambda_1(t) &\leq \{\beta_0 + (1 + \beta_0)(\sqrt{\delta})^{-1}\}\widehat{\xi}_2\mathbb{E}|Y_t^x - Y_{[t/\delta]\delta}^x|^2 \\ &\quad + \widehat{\xi}_2\sqrt{\delta}\mathbb{E}|b(\overline{Y}_{[t/\delta]\delta}^x, \Lambda_{[t/\delta]\delta}^i)|^2 + \beta_0\widehat{\xi}_2\sqrt{\delta}\mathbb{E}|Y_t^x|^2 \\ &\leq c + \{\beta_0 + 8L^2(5 + 4\beta_0)\}\widehat{\xi}_2\overline{\xi}_2\sqrt{\delta}\mathbb{E}(|Y_t^x|^2\xi_{\Lambda_t^i}^{(2)}).\end{aligned}$$

Next, by (3.13) with $p = 2$ and (3.23), we have

$$\Lambda_2(t) \leq 2\beta_0\widehat{\xi}_2\mathbb{E}(|Y_{[t/\delta]\delta}^{x,i}|^2\mathbf{1}_{\Lambda_t^i \neq \Lambda_{[t/\delta]\delta}^i}) \leq c + 8q_0\beta_0\widehat{\xi}_2\overline{\xi}_2\delta\mathbb{E}(|Y_t^{x,i}|^2\xi_{\Lambda_t^i}^{(2)}).$$

Thus, we arrive at

$$e^{\rho t}\mathbb{E}((1 + |Y_t^{x,i}|^2)^{p/2}\xi_{\Lambda_t^i}^{(2)}) \leq c(|x|^2 + e^{\rho t}) - \mathbb{E} \int_0^t e^{\rho s}(\eta_2 - \rho - \beta\sqrt{\delta})(1 + |Y_s^{x,i}|^2)^{p/2}\xi_{\Lambda_s^i}^{(2)}ds.$$

Taking $\rho = \eta_2 - \alpha\sqrt{\delta} > 0$ thanks to (3.21) yields the desired assertion (3.22).

(ii) Uniqueness of Invariant Measure. We need only to show that

$$\mathbb{E}(Y_t^{x,i} - Y_t^{y,i}|^2) \leq ce^{-\rho t}|x - y|^2. \quad (3.26)$$

For $\delta \in (0, 1)$ such that (3.21), we deduce from (3.2) and (3.3) that

$$\mathbb{E}|\overline{Y}_{[t/\delta]\delta}^{x,i} - \overline{Y}_{[t/\delta]\delta}^{y,i}|^2 \leq 6\mathbb{E}|Y_t^{x,i} - Y_t^{y,i}|^2, \quad (3.27)$$

and

$$\mathbb{E}|Y_t^{x,i} - Y_t^{y,i} - (\overline{Y}_{[t/\delta]\delta}^{x,i} - \overline{Y}_{[t/\delta]\delta}^{y,i})|^2 \leq 24L^2\delta\mathbb{E}|Y_t^{x,i} - Y_t^{y,i}|^2. \quad (3.28)$$

For any $\rho > 0$, by Itô's formula and (2.3), it follows from (3.19) that

$$\begin{aligned}&\mathbb{E}(e^{\rho t}|Y_t^{x,i} - Y_t^{y,i}|^2\xi_{\Lambda_t^i}^{(2)}) \\ &\leq |x - y|^2\xi_i^{(2)} - (\eta_2 - \rho) \int_0^t e^{\rho s}\mathbb{E}(|Y_s^{x,i} - Y_s^{y,i}|^2\xi_{\Lambda_s^i}^{(2)})ds + \int_0^t e^{\rho s}\Gamma(s)ds,\end{aligned}$$

where

$$\begin{aligned}\Gamma(t) &= 2\mathbb{E}(\langle Y_t^{x,i} - Y_t^{y,i} - (\overline{Y}_{[t/\delta]\delta}^{x,i} - \overline{Y}_{[t/\delta]\delta}^{y,i}), b(\overline{Y}_{[t/\delta]\delta}^{x,i}, \Lambda_{[t/\delta]\delta}^i) - b(\overline{Y}_{[t/\delta]\delta}^{y,i}, \Lambda_{[t/\delta]\delta}^i) \rangle \xi_{\Lambda_t^i}^{(2)}) \\ &\quad + 2\mathbb{E}(\langle Y_t^{x,i} - Y_t^{y,i}, \overline{Y}_{[t/\delta]\delta}^{x,i} - \overline{Y}_{[t/\delta]\delta}^{y,i} - (Y_t^x - Y_t^y) \rangle \beta_{\Lambda_t^i}\xi_{\Lambda_t^i}^{(2)}) \\ &\quad + \mathbb{E}(|\overline{Y}_{[t/\delta]\delta}^{x,i} - \overline{Y}_{[t/\delta]\delta}^{y,i} - (Y_t^{x,i} - Y_t^{y,i})|^2\beta_{\Lambda_t^i}\xi_{\Lambda_t^i}^{(2)}) \\ &\quad + \mathbb{E}((\beta_{\Lambda_{[t/\delta]\delta}^i} - \beta_{\Lambda_t^i})|\overline{Y}_{[t/\delta]\delta}^{x,i} - \overline{Y}_{[t/\delta]\delta}^{y,i}|^2\xi_{\Lambda_t^i}^{(2)}).\end{aligned}$$

Notice from (3.3), (3.27) and (3.28) that

$$\Gamma(t) \leq \beta\sqrt{\delta}\mathbb{E}(|Y_t^{x,i} - Y_t^{y,i}|^2\xi_{\Lambda_t^i}^{(2)}),$$

in which $\beta > 0$ is defined in (3.20). Consequently, we have

$$\mathbb{E}(e^{\rho t} |Y_t^{x,i} - Y_t^{y,i}|^2 \xi_{\Lambda_t^i}^{(2)}) \leq |x - y|^2 \xi_i^{(2)} - (\eta_2 - \rho - \beta\sqrt{\delta}) \int_0^t e^{\rho s} \mathbb{E}(|Y_s^{x,i} - Y_s^{y,i}|^2 \xi_{\Lambda_s^i}^{(2)}) ds,$$

Choosing $\rho = (\eta_2 - \beta\sqrt{\delta}) > 0$ due to (3.21) leads to (3.26). \square

Now we construct an example to show an application of Theorem 3.3.

Example 3.4. Let $\{\Lambda_t\}_{t \geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{1, 2, 3\}$ with the generator

$$Q = \begin{pmatrix} -(3 + \nu) & \nu & 3 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix}$$

for some $\nu \geq 0$. Consider a scalar linear SDE with regime switching

$$dX_t = \alpha_{\Lambda_t} X_t dt + \sigma_{\Lambda_t} X_t dW_t, \quad t \geq 0, \quad X_0 = x, \quad \Lambda_0 = i_0, \quad (3.29)$$

where $\alpha, \sigma : \mathbb{S} \mapsto \mathbb{R}$ such that

$$\alpha_1 = \frac{1}{2}, \alpha_2 = -2, \alpha_3 = -3, \quad \sigma_1 = \frac{1}{3}, \sigma_2 = 2, \sigma_3 = 1.$$

Observe that (3.3) holds with $L = 4$, and **(H)** holds for $\beta_1 = \frac{10}{9}, \beta_2 = 0$, and $\beta_3 = -5$. Since the Markov chain possesses the stationary distribution

$$\mu = (\mu_1, \mu_2, \mu_3) = \left(\frac{5}{20 + 5\nu}, \frac{6 + 3\nu}{20 + 5\nu}, \frac{9 + 2\nu}{20 + 5\nu} \right).$$

Note that the solution of the equation

$$dX_t^{(1)} = \alpha_{\Lambda_t} X_t^{(1)} dt + \sigma_{\Lambda_t} X_t^{(1)} dW_t$$

with $X_0^{(1)} \neq 0$ will explode to ∞ with probability one. However it is easy to see that (2.6) and (3.18) are satisfied respectively for any $\nu \geq 0$. Then, $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ has a unique invariant measure for sufficiently small $\delta \in (0, 1)$.

We can also obtain the following convergence rate of numerical invariant measure.

Theorem 3.5. Under the assumptions of Theorem 3.3, for $\delta \in (0, 1)$ such that (3.21), there exists $c > 0$ such that

$$W_1(\pi, \pi^\delta) \leq c\delta^{1/2}.$$

Proof. We omit the proof of Theorem 3.5 since it is similar to that of Theorem 3.2. \square

4 Numerical Invariant Measure: Reversible Case

In the last section, we investigate existence and uniqueness of numerical invariant measures for RSDPs with additive noises and multiplicative noises respectively, where the Markov chain $\{\Lambda_t\}_{t \geq 0}$ need not to be reversible, i.e., $\pi_i q_{ij} = \pi_j q_{ji}, i, j \in \mathbb{S}$, for some probability measure $\pi := (\pi_1, \dots, \pi_N)$. While, throughout this section, we shall always assume that the Markov chain $\{\Lambda_t\}_{t \geq 0}$, with the state space $\mathbb{S} := \{1, \dots, N\}$, $N < \infty$, is reversible with the probability measure π above. For such case, under a new condition we study existence and uniqueness of numerical invariant measure for multiplicative noise case.

To begin with, we need to introduce some notation. Let

$$L^2(\pi) := \left\{ f \in \mathcal{B}(\mathbb{S}) : \sum_{i=1}^N \pi_i f_i^2 < \infty \right\}.$$

Then $(L^2(\pi), \langle \cdot, \cdot \rangle_0, \|\cdot\|_0)$ is a Hilbert space with the inner product $\langle f, g \rangle_0 := \sum_{i=1}^N \pi_i f_i g_i, f, g \in L^2(\pi)$. Define the bilinear form $(D(f), \mathcal{D}(D))$ as

$$D(f) := \frac{1}{2} \sum_{i,j=1}^N \pi_i q_{ij} (f_j - f_i)^2 - \sum_{i=1}^N \pi_i \beta_i f_i^2, \quad f \in L^2(\pi),$$

where $\beta_i \in \mathbb{R}, i \in \mathbb{S}$, is given in **(H)**, and the domain

$$\mathcal{D}(D) := \{f \in L^2(\pi) : D(f) < \infty\}.$$

The principal eigenvalue λ_0 of $D(f)$ is defined by

$$\lambda_0 := \inf \{D(f) : f \in \mathcal{D}(D), \|f\|_0 = 1\}.$$

For more details on the first eigenvalue, refer to [7, Chapter 3]. Due to the fact that the state space of $\{\Lambda_t\}_{t \geq 0}$ is finite, there exists $\xi = (\xi_1, \dots, \xi_N) \in \mathcal{D}(D)$ such that

$$D(\xi) = \lambda_0 \|\xi\|_0^2. \quad (4.1)$$

For $\xi \in \mathcal{D}(D)$ such that (4.1) holds, set

$$\tilde{\xi}_1 := \max_{i \in \mathbb{S}} \xi_i, \quad \tilde{\xi}_2 := (\min_{i \in \mathbb{S}} \xi_i)^{-1}.$$

Let

$$\kappa := \{(1 + 12q_0)\beta_0 + 8L^2(5 + 6\beta_0)\} \tilde{\xi}_1 \tilde{\xi}_2, \quad (4.2)$$

where q_0, β_0 are given in (3.6), and $L > 0$ defined in (3.3).

The main result in this section is the following.

Theorem 4.1. Let $N < \infty$, (3.3) and **(H)** hold, and assume further $\lambda_0 > 0$. Then, $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ admits a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ for any

$$\delta < (1/(32L^2)) \wedge (\lambda_0/\kappa)^2.$$

Proof. Recalling (4.1) and checking the argument of [21, Theorem 3.2], one has

$$\xi \gg \mathbf{0} \quad \text{and} \quad (Q\xi)(i) + \beta_i \xi_i = -\lambda_0 \xi_i, \quad i \in \mathbb{S}.$$

The remainder of the proof is similar to that of Theorem 3.3, here we omit it. \square

Next, an example is constructed to demonstrate Theorem 4.1.

Example 4.2. Let $\{\Lambda_t\}_{t \geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0, 1, 2\}$ with the generator

$$Q = \begin{pmatrix} -b & b & 0 \\ 2a & -2(a+b) & 2b \\ 0 & 3a & -3a \end{pmatrix}$$

for some $a, b > 0$. Consider a scalar SDE with regime switching

$$dX_t = \alpha_{\Lambda_t} X_t dt + \sigma_{\Lambda_t} X_t dW_t, \quad t \geq 0, \quad X_0 = x, \quad (4.3)$$

where $\alpha, \sigma : \mathbb{S} \mapsto \mathbb{R}$ such that

$$c_0 = 2\alpha_0 + \sigma_0^2 < 0, \quad c_1 = 2\alpha_1 + \sigma_1^2, \quad c_2 = 2\alpha_2 + \sigma_2^2.$$

We further assume that

$$b + c_0 < 0, \quad a - b - c_1 > 0, \quad a - c_2 > 0. \quad (4.4)$$

Note that (3.3) holds with $L = \max_{i \in \mathbb{S}} \{|\alpha_i| + |\sigma_i|\}$ and **(H)** holds with

$$\beta_0 = c_0, \quad \beta_1 = c_1, \quad \beta_2 = c_2.$$

Set

$$\Omega := Q + \text{diag}(\beta_1, \dots, \beta_N).$$

By the notion of Ω , for $\xi_i = i + 1$, $i = 0, 1, 2$, we deduce that

$$(\Omega\xi)(0) = -(-b - c_0)\xi_0, \quad (\Omega\xi)(1) = -(a - b - c_1)\xi_1, \quad (\Omega\xi)(2) = -(a - c_2)\xi_2.$$

Taking

$$\lambda = \min\{-b - c_0, a - b - c_1, a - c_2\} > 0$$

thanks to (4.4), one finds that

$$(\Omega\xi)(i) \leq -\lambda \xi_i, \quad i = 0, 1, 2.$$

Then $\lambda_0 > 0$ due to [21, Theorem 4.4]. As a result, $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ has a unique invariant measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ whenever the stepsize is sufficiently small.

Remark 4.1. The principal-eigenvalue approach has been applied successfully to investigate ergodic property, stability and recurrence for regime-switching diffusion processes. For more details, please refer to Shao [22] and Shao-Xi [23]. As we discuss previously, for the reversible case, such trick can also be utilized to discuss existence and uniqueness of numerical invariant measure for RSDPs with multiplicative noises.

Remark 4.2. Theorem 4.1 can also be extended into the case of RSDPs with countable state spaces (i.e. $N = \infty$) provided that λ_0 is attainable, i.e., there exists $f \in L^2(\pi)$, $f \neq 0$, such that $D(f) = \lambda_0 \|f\|_0^2$. For more details, please refer to [21, Theorem 3.2].

5 Numerical Invariant Measure: Countable State Space

The approach based on the Perron-Frobenius theorem (see Theorem 3.1 and 3.3) is not suitable to the case that \mathbb{S} is a countable state space, i.e., $N = \infty$, while the approach based on the principal eigenvalue (see Theorem 4.1) can be applied to this case under some additional conditions as being pointed out in Remark 4.2. Now in this section, we shall introduce another method to deal with the case $N = \infty$, based on a finite partition approach and an M -matrix theory.

Definition 5.1. (see e.g. [15, Definition 2.9, p.67]) A square matrix $A = (a_{ij})_{n \times n}$ is called a nonsingular M -matrix if A can be expressed in the form $A = sI - B$ with $B \gg \mathbf{0}$ and $s > \text{Ria}(B)$, where I is the $n \times n$ identity matrix and $\text{Ria}(B)$ the spectral radius of B .

We further suppose that

$$K := \sup_{i \in \mathbb{S}} \beta_i < \infty \quad \text{and} \quad \sup_{i \in \mathbb{S}} (-q_{ii}) < \infty, \quad (5.1)$$

where $\beta_i \in \mathbb{R}$ is given in **(H)**. Let us insert m points in the interval $(-\infty, K]$ as follows:

$$-\infty =: k_0 < k_1 < \cdots < k_m < k_{m+1} := K.$$

Then, the interval $(-\infty, K]$ is divided into $m + 1$ sub-intervals $(k_{i-1}, k_i]$ indexed by i . Let

$$F_i := \{j \in \mathbb{S} : \beta_j \in (k_{i-1}, k_i]\}, \quad i = 1, \dots, m + 1.$$

Without loss of generality, we can and do assume that each F_i is not empty. Then

$$F := \{F_1, \dots, F_{m+1}\}$$

is a finite partition of \mathbb{S} . For $i, j = 1, \dots, m + 1$, set

$$q_{ij}^F := \begin{cases} \sup_{r \in F_i} \sum_{k \in F_j} q_{rk}, & j < i, \\ \inf_{r \in F_i} \sum_{k \in F_j} q_{rk}, & j > i, \\ -\sum_{j \neq i} q_{ij}^F, & i = j. \end{cases}$$

So $Q^F := (q_{ij}^F)$ is the Q -matrix for some Markov chain with the state space $\mathbb{S}_0 := \{1, \dots, m + 1\}$. For $i = 1, \dots, m + 1$, let

$$\beta_i^F := \sup_{j \in F_i} \beta_j, \quad H_{m+1} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(m+1) \times (m+1)}.$$

Theorem 5.1. Let $N = \infty$, (3.3), **(H)** and (5.1) hold. Assume further that $\{\Lambda_t\}_{t \geq 0}$ is exponential ergodic and that

$$-(Q^F + \text{diag}(\beta_1^F, \dots, \beta_{m+1}^F))H_{m+1}$$

is a nonsingular M -matrix. Then $(\overline{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ admits a unique measure $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ whenever the stepsize $\delta \in (0, 1)$ is sufficiently small.

Proof. Some ideas of the argument go back to [22, Theorem 4.1]. Moreover, we only sketch the argument of Theorem 5.1 since it is analogous to that of Theorem 3.3.

Since $-(Q^F + \text{diag}(\beta_1^F, \dots, \beta_{m+1}^F))H_{m+1}$ is a nonsingular M -matrix, by [15, Theorem 2.10, p.68] there exists a vector $\eta^F := (\eta_1^F, \dots, \eta_{m+1}^F)^* \gg \mathbf{0}$ such that

$$(-\lambda_1^F, \dots, -\lambda_{m+1}^F)^* := (Q^F + \text{diag}(\beta_1^F, \dots, \beta_{m+1}^F))H_{m+1}\eta^F \ll \mathbf{0}. \quad (5.2)$$

Set $\xi^F := H_{m+1}\eta^F$. By the structure of H_{m+1} , it is trivial to see that

$$\xi_i^F = \eta_{m+1}^F + \dots + \eta_i^F, \quad i = 1, \dots, m+1.$$

This, together with $\eta^F \gg \mathbf{0}$, yields that $\xi^F \gg \mathbf{0}$ and $\xi_{i+1}^F < \xi_i^F, i = 1, \dots, m+1$. Next, we extend the vector ξ^F to be a vector on \mathbb{S} by setting $\xi_r := \xi_i^F$ for $r \in F_i$. Moreover, let $\phi : \mathbb{S} \mapsto \{1, \dots, m+1\}$ be a map defined by $\phi(j) := i$ for $j \in F_i$. Then, by the definition of β_i^F , one has

$$\xi_r = \xi_i^F = \xi_{\phi(r)}^F \quad \text{and} \quad \beta_r \leq \beta_{\phi(r)}^F, \quad r \in F_i. \quad (5.3)$$

For any $r \in \mathbb{S}$, there exists F_i such that $r \in F_i$. Recalling the definition of q_{ij}^F and utilizing $\xi_{i+1}^F < \xi_i^F, i = 1, \dots, m+1$, we derive from (5.3) that, for $r \in F_i$,

$$\begin{aligned} (Q\xi)(r) &= \sum_{k < i} \sum_{j \in F_k} q_{rj}(\xi_j - \xi_r) + \sum_{k > i} \sum_{j \in F_k} q_{rj}(\xi_j - \xi_r) \\ &= \sum_{k < i} \sum_{j \in F_k} q_{rj}(\xi_k^F - \xi_i^F) + \sum_{k > i} \sum_{j \in F_k} q_{rj}(\xi_k^F - \xi_i^F) \\ &\leq \sum_{k < i} q_{ik}^F(\xi_k^F - \xi_i^F) + \sum_{k > i} q_{ik}^F(\xi_k^F - \xi_i^F) \\ &= (Q^F \xi^F)(i) = (Q^F \xi^F)(\phi(r)). \end{aligned} \quad (5.4)$$

For any $\rho > 0$, observe from (5.2)-(5.4) that

$$\begin{aligned}
& e^{\rho t} \mathbb{E}(|Y_t^x|^2 \xi_{\Lambda_t^i}) \\
& \leq c(|x|^2 + e^{\rho t}) + \mathbb{E} \int_0^t e^{\rho s} \{ \rho \xi_{\Lambda_s^i} + (Q\xi)(\Lambda_s^i) + \beta_{\Lambda_s^i} \xi_{\Lambda_s^i} \} |Y_s^x|^2 ds + \int_0^t e^{\rho s} \{ \Lambda_1(s) + \Lambda_2(s) \} ds \\
& \leq c(|x|^2 + e^{\rho t}) + \mathbb{E} \int_0^t e^{\rho s} \{ \rho \xi_{\phi(\Lambda_s^i)}^F + (Q^F \xi^F)(\phi(\Lambda_s^i)) + \beta_{\phi(\Lambda_s^i)}^F \xi_{\phi(\Lambda_s^i)}^F \} |Y_s^x|^2 ds \\
& \quad + \int_0^t e^{\rho s} \{ \Lambda_1(s) + \Lambda_2(s) \} ds \\
& \leq c(|x|^2 + e^{\rho t}) - \mathbb{E} \int_0^t e^{\rho s} \{ \lambda_{\phi(\Lambda_s^i)}^F - \rho \xi_{\phi(\Lambda_s^i)}^F \} |Y_s^x|^2 ds + \int_0^t e^{\rho s} \{ \Lambda_1(s) + \Lambda_2(s) \} ds \\
& \leq c(|x|^2 + e^{\rho t}) - (\lambda_0 - \rho \xi_0) \mathbb{E} \int_0^t e^{\rho s} |Y_s^x|^2 ds + \int_0^t e^{\rho s} \{ \Lambda_1(s) + \Lambda_2(s) \} ds
\end{aligned}$$

where $\lambda_0 := \min_{i \in \mathbb{S}_0} \lambda_i^F > 0$ due to (5.2), $\xi_0 := \max_{i \in \mathbb{S}_0} \xi_i^F$, and Λ_i is defined as in (3.25). Then, following the argument of Theorem 3.3, we have

$$\sup_{t \geq 0} \mathbb{E} |Y_t^x|^2 < \infty.$$

Then, ergodicity of $\{\Lambda_t\}_{t \geq 0}$ yields existence of numerical invariant measure whenever the stepsize $\delta > 0$ is sufficiently small. The proof of the uniqueness is similar to that of Theorem 5.1, since Markov chain $\{\Lambda_t\}_{t \geq 0}$ is exponential ergodic, (2.10) holds. \square

Remark 5.1. $(\bar{Y}_{k\delta}^{x,i}, \Lambda_{k\delta}^i)$ associated with [22, Example 4.1] admits a numerical invariant measure whenever the stepsize $\delta > 0$ is sufficiently small.

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